An SU(3) Casson invariant for Rational Homology Spheres

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Historical Background

- 1990 Casson defined $\lambda : \{\mathbb{Z} \text{ homology spheres}\} \to \mathbb{Z}$ using $Hom(\pi_1 X, SU(2))$
- 1992 Taubes reinterpreted λ using $SU(2)$ gauge theory
- 1988 Floer defined instanton homology groups $IFH(X)$; Taubes showed $\chi(IFH(X^3)) = \lambda(X)$
- Walker extended λ to $\mathbb{Q}$HS's; Lescop extended it to all 3-manifolds
- 1998 Boden-H. gave $SU(3)$ generalization $\lambda_{SU(3)}$ for $\mathbb{Z}$HS's
  2001 Boden-H.-Kirk "renormalized" invariant $\tau_{SU(3)}$ to obtain simpler, integer-valued invariant
  2002 Cappell-Lee-Miller gave an alternative renormalization
- 2005 BHK calculated $\tau_{SU(3)}$ for $\mathbb{Z}$HS surgeries on torus knots and other Brieskorn homology spheres

Motivation for current work

For $1/n$ surgery on (p,q)-torus knot,

$$\tau_{SU(3)} = B(p,q)n + C(p,q)n^2,$$

where $C(p,q)$ is one Conway polynomial coefficient for the knot. $B(p,q)$ is not recognizable. Calculations were also done for Seifert fibered homology spheres $\Sigma(p,q,r)$.

$\tau_{SU(3)}$ is not finite type, but perhaps it differs from a finite type invariant by something we can recognize.

Motivation for current work

Extending $\tau_{SU(3)}$ to $\mathbb{Q}$HSs will give more families of Seifert fibered manifolds, related by overlapping surgery sequences, for which we can do similar calculations. This may help uncover a conjectural pattern.

Work in progress with Boden and Himpel computes $\tau_{SU(3)}$ for other splice sums involving torus knot complements (or more generally certain Seifert fibered knot complements). Our methods for computing moduli spaces and spectral flow would work just as well on rational homology spheres.

Morse theory analogy

Theorem

For any Morse function $f : M^n \to \mathbb{R}$ on a closed manifold, $\sum_{c \in crit(f)} (-1)^{\mu(c)}$ equals $\chi(M)$, the Euler characteristic. Here, $\mu(c)$ denotes the Morse index of the critical point c.

For direct proof of independence of f:

For a generic path from f_0 to f_1, the parameterized critical set $W = \{(x, t) \mid x \in crit(f_t)\}$ gives cobordism between $crit(f_0)$ and $crit(f_1)$.

Gauge theory analog

$\mathcal{A} = \{SU(n)$ connections on $E = X^3 \times \mathbb{C}^n\}$.
$cs : \mathcal{A} \to \mathbb{R}$ is the Chern-Simons function.
Or $cs + h : \mathcal{A} \to \mathbb{R}$, where h is a holonomy perturbation.
Critical points of cs are flat connections.
(These are related to homomorphisms from $\pi_1(X)$ to $SU(n)$.)

One potential problem (in two guises)

Gauge group $\mathcal{G} = Aut(E)$ acts on $\mathcal{A}$ with varying orbit types. The reducible connections $A = A_1 \oplus \cdots \oplus A_k$ on splitting $E = E_1 \oplus \cdots \oplus E_k$ have larger isotropy.

Equivariance Prevents Transversality to Get Cobordism

$cs : \mathcal{A} \to \mathbb{R}$ is $\mathcal{G}$ invariant, so $crit(cs)$ consists of whole orbits, not isolated connections. Worse than that, gauge invariance of $cs + h$ prevents transversality needed to get cobordism and invariance under perturbation.

Quotient Singularities

Consider $cs : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$.
The critical set of cs on the quotient equals $\mathcal{M} = \{\text{flat connections}\}/\mathcal{G}$.
But $\mathcal{B} = \mathcal{A}/\mathcal{G}$ has singularities at the orbits of reducible connections, and now singularities interfere with cobordism arguments.

Singularities interfere with cobordism arguments

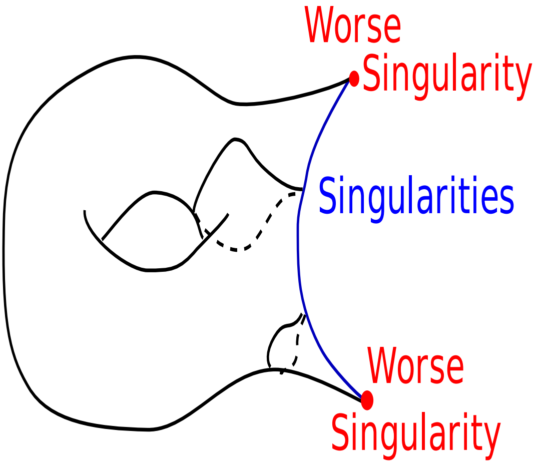

Figure: A path of functions on a singular manifold does not produce a cobordism between critical sets. Critical points disappear into singular strata (and critical points in one stratum disappear into another stratum).

Homology restrictions

Homology restrictions on X limit abelian representations of $\pi_1(X)$. Limiting the rank n of the group $SU(n)$ limits the types of reductions.

Simplest case: $SU(2)$, $H_*(X;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$

$\mathcal{M} = \mathcal{M}^* \cup \{[\theta]\}$, where θ =trivial connection, and
$\mathcal{M}^* = \{\text{irreducibles}\}$ is compact.

Theorem (Taubes)

$$\sum_{\substack{[A] \in crit(cs+h) \\ [A] \neq [\theta]}} (-1)^{SF(\theta,A)} = \lambda_{SU(2)}(X)$$

Generalizations dealing with reducibles

Walker extended $\lambda_{SU(2)}$ to $\mathbb{Q}$-homology spheres, i.e., X^3 such that $H_*(X;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$. It counts irreducibles with sign, and adds a correction term involving *abelians*, so the combination is perturbation invariant.

[Boden-H., B.-H.-Kirk]
For $SU(3)$ and $\mathbb{Z}$ homology spheres, $\mathcal{M} = \mathcal{M}^* \cup \mathcal{M}^{red} \cup \{[\theta]\}$, but $\{[\theta]\}$ is isolated. A correction term involving reducibles of the form $A = A_1 \oplus A_2$ compensates for the dependence of

$$\sum_{[A]\in crit(cs+h) \text{ irred}} (-1)^{SF(\theta,A)}$$

on the perturbation.

Correction term details for $2 \oplus 1$ connections

SF denotes spectral flow of the twisted signature operator, which amounts to a relative Morse index between two critical points in this infinite dimensional context.

$$\tau_{SU(3)}(X) = \sum_{[A]\in\mathcal{M}_h^*} (-1)^{SF(\theta,A)} - \sum_{[B]\in\mathcal{M}_h^{red}} (-1)^{SF(\theta,B)} \left[\frac{SF_N(B_0,B)}{2}\right].$$

Decompose spectral flow (along a path of reducible connections) into "tangent to reducibles" and "normal to reducibles" components. SF_N is the latter.

dishonest notation

B_0 in this formula stands for a $[B]$-dependent reducible basepoint in the component of moduli space $\mathcal{M}^{red}$ from which $B \in \mathcal{M}_h^{red}$ arises.

Properties of $\tau_{SU(3)}$ on $\mathbb{Z}$HSs

- $\tau_{SU(3)}(X) \in \mathbb{Z}$.
- $\tau_{SU(3)}(-X) = \tau_{SU(3)}(X)$.
- $\tau_{SU(3)} - 2\lambda^2_{SU(2)}$ is additive under connected sum.
- $\tau_{SU(3)}$ is a sum well-defined contributions from the components of the flat moduli space.
  (In contrast, CLM invariant is a sum of a perturbation dependent count of $\mathcal{M}^{ir}$, spectral flows to the reducibles, and rank of $SU(2)$ Floer boundary operator.)

In choosing "basepoints" from whence to measure spectral flow to perturbed flat connections, there are various choices. These properties guide the choices.

Extending $\tau_{SU(3)}$ to $\mathbb{Q}$HS case

(joint work in progress with Hans Boden)
Consider $SU(3)$ connections on a rational homology sphere X.

Orbit types (i.e., singular strata in $\mathcal{A}/\mathcal{G}$):

3	irreducible,
$2 \oplus 1$	$A_1 \oplus A_2$, of ranks 2 and 1
$1 \oplus 1 \oplus 1$	$A_1 \oplus A_2 \oplus A_3$, distinct rank 1 connections
$1 \oplus 1^2$	$A_1 \oplus A_2 \oplus A_2$
1^3	central

Bifurcations and the parameterized moduli space

Question

What effect do these different types of singularities play?

Specifically, $\mathcal{A}/\mathcal{G}$ is stratified by these 5 orbit types.

Question

Besides "birth/death of cancelling pairs" within each stratum, how can the critical set change?

We keep track of changes in topology of the (perturbed) flat moduli space by working with the *parameterized moduli space*,
$W = \{([A], t) \in \mathcal{B} \mid grad(cs + h_t)(A) = 0\}$, for any path $h_t, 0 \leq t \leq 1$.

Structure of the parameterized moduli space

The structure of $W = \{([A], t) \in \mathcal{B} \mid grad(cs + h_t)(A) = 0\}$ for a generic path $h_t, 0 \le t \le 1$, $SU(3)$, and $\mathbb{Q}$HS, is described in *Transversality for equivariant exact 1-forms and gauge theory on 3 manifolds*, H., AIM 2006.

$\mathbb{Q}$HS restriction implies abelians are isolated from one another. This shows $W^{1\oplus 1\oplus 1}$, $W^{1^2\oplus 1}$ and W^{1^3} form compact product cobordisms.

$W^{2\oplus 1}$ is a compact cobordism except for ends hitting $W^{1\oplus 1\oplus 1}$ and $W^{1^2\oplus 1}$.

W^* is compact except for ends hitting $W^{2\oplus 1}$.

Structure of the parameterized moduli space

The ‘cobordism’ connecting two critical sets will have 3 types of singularities, all modeled on T-intersections.

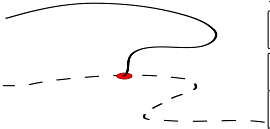

Figure: A T-intersection.

Levels of reducibility

Solid	Dashed	Description
Irreducible	$2 \oplus 1$	W^* runs into $W^{2\oplus 1}$
$2 \oplus 1$	$1 \oplus 1 \oplus 1$	$W^{2\oplus 1}$ runs into $W^{1\oplus 1\oplus 1}$
$2 \oplus 1$	$1^2 \oplus 1$	$W^{2\oplus 1}$ runs into $W^{1^2\oplus 1}$

Key Result

The spectral flow between any three endpoints is determined by normal spectral flow across T-intersection.

Correction terms to account for bifurcations

Birth of Irreducibles

Irreducible critical points can pop out of $2 \oplus 1$ critical points. $\sum_{\mathcal{M}_h^*} (-1)^{SF(\theta,A)}$ changes by ± 1 when irreducibles can pop out of reducibles, as perturbation is varied. We need the [BHK] correction term for $2 \oplus 1$ stratum.

Birth or Death of Next Level Reducibles

In addition, $2 \oplus 1$ critical points can pop out of the $1 \oplus 1 \oplus 1$ or $1 \oplus 1^2$ strata. The [BHK] correction term changes (in a different way than adding ± 1) when this happens, so we need secondary $1 \oplus 1 \oplus 1$ and $1 \oplus 1^2$ correction terms that account for this.

Normal spectral flow along abelians

Consider a path of abelian connections $C(t) = C_1(t) \oplus C_2(t) \oplus C_3(t)$ on $E = E_1 \oplus E_2 \oplus E_3$. Stabilizer is a 2-torus.

$T_{C(t)}\{\text{abelian connections}\} = \Omega^1(X; diag(su(3))$.

Normal bundle is $\Omega^1(X; \mathbb{C}^3)$ where

$$\mathbb{C}^3 = \left\{ \begin{bmatrix} 0 & z_{12} & z_{13} \\ -\overline{z}_{12} & 0 & z_{23} \\ -\overline{z}_{13} & -\overline{z}_{23} & 0 \end{bmatrix} \middle| (z_{12}, z_{13}, z_{23}) \in \mathbb{C}^3 \right\}$$

and the different $\mathbb{C}_{ij}$ summands have different weights.

Spectral flow in $\mathbb{C}_{ij}$ occurs whenever a $2 \oplus 1$ flat connection intertwining i, j line bundles pops out or dies.

Abelian correction terms

Idea

$$\frac{1}{4}\sum_{[C]\in\mathcal{M}^{1\oplus1\oplus1}}(-1)^{SF(\theta,C)}\left[SF_{12}(C_0,C)SF_{13}(C_0,C)\right.$$
$$\left.SF_{12}(C_0,C)SF_{23}(C_0,C)+SF_{13}(C_0,C)SF_{23}(C_0,C)\right].$$

More precisely, for each $2\oplus1$ splitting $E=M_{-\omega}\oplus L_\omega$, each $C=(C_1\oplus C_2)\oplus C_3$ on this splitting contributes

$$\tfrac{1}{4}(-1)^{SF(\theta,C)}\left[SF_{12}(C_0,C)SF_{N,\omega}(B_0,C)\right]$$

where $SF_{N,\omega}$ means normal to $M_{-\omega}\oplus L_\omega$ reducibles, and B_0 is the chosen $2+1$ basepoint near C and C_0 is the nearby abelian flat connection.

$[C_1\oplus C_2\oplus C_3]\in\mathcal{M}_h^{1\oplus1\oplus1}$ contributes to three different $2\oplus1$ splittings.

Right way to organize this

On a $\mathbb{Q}$HS X, there are many splittings of $X \times \mathbb{C}^3$ into $M^{(2)}_{-\omega} \oplus L^{(1)}_{\omega}$ indexed by $\omega \in H^2(X;\mathbb{Z})$.

Define $2 \oplus 1$ correction terms and abelian correction terms for each ω, and for each component of

$$\mathcal{M}_\omega = \left(\mathcal{M}^{2\oplus 1} \cup \mathcal{M}^{1\oplus 1\oplus 1} \cup \mathcal{M}^{1\oplus 1^2}\right) \cap \left\{\text{conn's compatible w/ } M^{(2)}_{-\omega} \oplus L^{(1)}_{\omega}\right\}$$

Select basepoint B^i_ω in the ith component of $\mathcal{M}_\omega$. $SF_{N,\omega}(B^i_\omega, B)$ from ith base point to any nearby $B \in \mathcal{M}_{h,\omega}$ is well-defined, whether it is $2 \oplus 1$ or abelian.

$$
\begin{aligned}
\tau_{SU(3)}(X) &= \sum_{[A]\in\mathcal{M}_h^*(X)} (-1)^{SF(\theta,A)} \\
&+ \sum_{\omega\in H^2(X;\mathbb{Z})} \Bigg(- \sum_{[B]\in\mathcal{M}_{h,\omega}^{2\oplus 1}} (-1)^{SF(\theta,B)} \left[\frac{SF_{N,\omega}(B_\omega^i, B)}{2} \right] \\
&+ \frac{1}{4} \sum_{[C]\in\mathcal{M}_{h,\omega}^{1\oplus 1\oplus 1}(X)\cup\mathcal{M}_{h,\omega}^{1\oplus 1^2}(X)} (-1)^{SF(\theta,C)} \left[SF_{12}(C_0, C) SF_{N,\omega}(B_\omega^i, C) \right] \Bigg).
\end{aligned}
$$

Theorem

With this formula, $\tau_{SU(3)}$ is integer valued, independent of perturbation, and independent of orientation on X.